YESHIVA UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICAL SCIENCES
TOPICS FOR THE PHD QUALIFYING EXAMINATION

The qualifying examination in mathematical sciences covers three areas:

(I) Real Analysis

(II) Complex Analysis

(III) Research Area

For the first two areas, a list of topics is below. Also, a list of sample exercises for the two areas is provided. The actual exercises asked on the exam will be different from the sample exercises; being able to solve the sample exercises is not sufficient for the exam preparation.

The third exam area pertains to the research subject that the student intends

7. Riemann-Lebesgue Theorem (outline). A function $f : [a; b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and its set of discontinuity points is of measure zero.

8. Sequences of functions (uniform convergence, properties, equi-continuity for a family of functions, Ascoli-Arzela's theorem).

Complex Analysis:

1. If f is complex differentiable at z then the Cauchy-Riemann equations are satisfied at z.

2. If the partial derivatives of u and v exist and are continuous at $(x; y)$ and the Cauchy-Riemann equations are satisfied then $f(z) = u(x; y) + iv(x; y)$ is complex differentiable at $z = x + iy$.

3. If $f'(z) = 0$ in a region D then f is constant on D.

4. If $|f(z)| < M$ on a curve C then $\left|\int_C f(z)dz\right| < ML$ where L ia the length of the curve.

5. The following statements are equivalent:

   (i) f has an antiderivative F;
   (ii) $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$;
   (iii) If C is a closed curve then $\int_C f(z)dz = 0$.

6. Cauchy-Goursat Theorem (outline). If f is analytic on and inside a simple closed curve C then $\int_C f(z)dz = 0$.

7. If f is analytic in the region between closed curves C_2 and C_1 with C_1 inside C_2 then
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

8. The Cauchy Integral Formula.

9. A bounded entire function is constant.

10. If f is analytic on annulus, it equals its Laurent series (outline).

11. Cauchy Residue Theorem.

Sample Exercises:

1. Let

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right); & \text{for } x \neq 0; \\ \ ; & \text{for } x = 0, \end{cases}$$

where $\ \in [\ 1; 1]$.

(a) Is f continuous?

(b) Does f have the intermediate value property?

2. Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right); & \text{for } x \neq 0; \\ 0; & \text{for } x = 0, \end{cases}$$

(a) Show that f is differentiable everywhere.

(b) Is f' continuous?

3. Given that f is a quadratic polynomial

$$f(x) = Kx^2 +$$

6. On uniform convergence.

   (a) Show that if a sequence $\{f_n\}$ of continuous functions converges uniformly on a domain $\Omega \subset \mathbb{R}$ to a function f, then the limit f is also continuous on Ω.

   (b) Let $\{f_n\}$ be a sequence of continuously differentiable functions such that $\{f_n\}$ and $\{f_n'\}$ converge uniformly on a domain Ω to the limiting functions f and g, respectively. Show that for every x in the interior of Ω,

$$g(x) \equiv \lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n(x)\right)' \equiv f'(x).$$

7. State Ascoli-Arzela's Theorem and outline its proof.

8. The sequence of continuous functions $\{f_n : [0, 2\pi] \to \mathbb{R}\}_{n\in\mathbb{N}}$ with f_n given by $f_n(x) = \sin(nx)$ is uniformly bounded, but not equicontinuous. Give an intuitive reason why such a sequence is not equicontinuous, then give a rigorous proof.

9. Compute the following integral limit

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n\sin(x/n)}{x(x^2+1)}\,dx$$

10. Consider the function

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & \text{if } z \neq 0; \\ 0, & \text{if } z = 0;. \end{cases}$$

   Is this function differentiable at $z = 0$? Is it continuous at

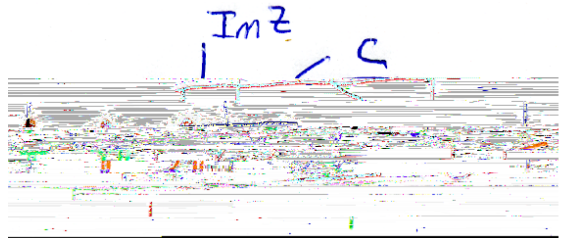

Figure 1: Contour C

15. Show that the only conformal maps from the complex plane onto itself are the non-constant linear maps, i.e. maps of the form $f(z) = az + b$, $a \neq 0$.

16. Let f be a *doubly periodic function*, that is, there are two complex numbers $w_1; w_2$ with $w_1/w_2 \notin \mathbb{R}$ so that for any $z \in \mathbb{C}$, $f(z) = f(z+w_1) = f(z+w_2)$. Let us also assume that f is meromorphic.

    (a) Show that if f is an entire function, then it has to be constant.

    (b) Let Γ be the boundary of the parallelogram with vertices $0; w_1; w_2; w_1 + w_2$, oriented counterclockwise. Show that if f is analytic on Γ, then $\int_\Gamma f(z)dz = 0$.

    (c) Assuming that f is analytic on Γ and has exactly one singularity inside Γ, show that the residue at this singularity is necessarily zero.

Bibliography:

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